Western Management Science Institute
University of California • Los Angeles

University of California

Los Angeles

Western Management Science Institute

Working Paper No. 83

The Economics of Uncertainty X

by

Karl Borch

June 1965

This paper consists of notes prepared for a series of lectures given at the Graduate School of Business Administration, at the University of California, Los Angeles, during the academic year 1964-1965. The work has been supported by the Western Management Science Institute under a grant from the Ford Foundation, and by the Office of Naval Research under Task 047-041. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Chapter X

Notes on Some Bargaining Problems

10.1 In Chapter VII we discussed a number of situations in which two persons could gain by making some kind of cooperative arrangement. We found that the set of all possible arrangements contained a subset of arrangements having the property which we called Pareto optimality. We concluded that two rational persons would somehow be able to reach a Pareto optimal arrangement. We were, however, not able to reach any firm conclusions as to which particular arrangement in the subset the two persons would make. All we had to say was that they in some way had to bargain until they agreed on a Pareto optimal arrangement. If our two persons were unable to do this, we would not consider them as "rational."

In this Chapter we shall study more general situations of this kind. Classical economic theory has little to say about the problems we want to analyse. We shall, however, see that the Theory of Games [13] makes it possible for us to come to grips with the problems, even if the theory at the present stage of development does not always yield satisfactory solutions.

10.2 In general an n-person game is described by the following three elements:

(i) A set N of n players.

(ii) n sets of strategies $S_1, S_2 \dots S_n$

The set S_i consists of the strategies $s_{i1}, s_{i2} \dots$ available to player i.

(iii) n payoff functions $M_1, M_2 \dots M_n$

The function $M_i = M_i (s_{1r_1}, s_{2r_2} \dots s_{nr_n})$

is the payoff to player i if

player 1 uses strategy $s_{1r_1} \in S_1$

player 2 uses strategy $s_{2r_2} \in S_2$

- -

If the rules of the game are such that each player must choose his strategy - pure or mixed - without any possibility of coordinating his choice with the choices of other players, we have a Non-cooperative game.

If the players have some possibilities of coordinating their choices - to their mutual advantage - we have a Cooperative game.

From this it follows that a complete description of the game must specify - in addition to the three basic elements - the possibilities of communication among the players.

For n=2 and $M_1(s_{1.}s_{2.}) = -M_2(s_{1.}s_{2.})$ we obtain the Two-person zero-sum game discussed in Chapter IX. In this case communication possibilities are irrelevant, since the players cannot both gain by coordinating their actions.

10.3 The model becomes very rich as soon as we drop the zero-sum condition, and it may be useful to discuss a few simple examples which will illustrate the wide variety of real life situations which can be represented as a game.

Let us consider two competing firms, and assume:

(i) If both firms maintain their selling price, each will make a profit of 1.

(ii) If one firm cuts the price, it will double its profits, provided

that the other firm maintains its price. The latter firm will then suffer a loss of 2.

(iii) If both firms cut the price, they will both lose 1.

This is an almost classical problem, usually known as "The Prisoner's Dilemma" ([10] p. 94). It is easy to see that the situation can be represented by the following payoff matrix:

		Firm 2: Maintain Price	Firm 2: Cut Price
Firm 1	Maintain Price	(1, 1)	(-2, 2)
	Cut price	(2, -2)	(-1, -1)

As a cooperative game this situation is trivial. The obvious "solution" is that the firms should agree to maintain the price. It seems, however, that they cannot reach this solution unless they can communicate with each other, and unless the agreement they reach must be observed by both parties.

10.4 If the two parties cannot communicate and make an enforceable agreement, the situation is far from trivial. To illustrate this, let us take the approach we used for the zero-sum game, and assume that the two firms maintain the price with probabilities x and y respectively. This will give the payoffs

$$M_1 (x, y) = -x + 3y - 1$$

$$M_2 (x, y) = 3x - y - 1$$

It is easy to see that there is no strategy which can secure any player an expected gain which is independent of what the opponent does. It is, however, clear that Firm 1 which controls x must choose $x = 0$

in order to make M_1 as large as possible, and that Firm 2 must choose $y = 0$. This means of course that both firms will cut price and suffer a loss.

It does not seem promising to continue analysing this situation in its full generality. If we ask a person how he would decide in a situation which can be represented by our model, it is most likely that he will reply that "it all depends."

10.5 This naturally leads us to examine the elements on which the decision may depend, for instance:

(i) The possibilities of communication.

(ii) The degree to which the two parties trust each other.

(iii) The magnitude of the payoffs.

(iv) The number of times the game will be played.

We can do this by introspection and theoretical arguments, and we can also do it by controlled experiments. The latter approach has been taken by Lave [8] and [9], and has led to some interesting results.

Lave studied a payoff matrix of the form

		Player 2	
		1	2
Player 1	1	(a, a)	(b, c)
	2	(c, b)	(d, d)

where $c > a > d > b$

He found that if the number of plays n was so large that

$$k\ (d - b) < n\ (a - d)$$

the subjects tended to choose the cooperative decision. In this inequality

k is a parameter which summarizes the attitudes of the subjects. $(d - b)$ represents the loss one suffers if one tries to cooperate, and the opponent does not respond. $(a - d)$ represents the gain obtained by cooperation. In his experiments with undergraduates at Reed College Lave found approximately $k = 3$. In a later experiment with Harvard undergraduates he found a stronger tendency to cooperate, regardless of the magnitude of the payoffs.

Experiments of this kind have a considerable psychological interest, but their significance for economics is questionable. The behavior of a student playing for pennies does not contribute much to our knowledge of economic behavior. It does not help if the experimenter asks the student to behave as if he were the president of U.S. Steel and had to make decisions involving millions of dollars. This will in Lave's words only give us information about "a subject's individual (almost certainly naive) conception of the way he thinks Roger Blough behaves." To a psychologist this may be interesting, but an economist may well dismiss such information as irrelevant to his problems.

10.6 As another example we shall discuss a second classical game, represented by the payoff matrix

		Player 2: 1	Player 2: 2
Player 1	1	(10, 5)	(0, 0)
	2	(0, 0)	(5, 10)

This game is generally known as the "Battle of the Sexes" ([10] p. 90), but it can easily be given economic interpretations.

Here again the first step towards a cooperative solution is trivial.

If the players can communicate, it seems obvious that they should agree to use either the two first strategies or the two second strategies, i.e. the "strategy pairs" (1, 1) or (2, 2). The two other strategy pairs (1, 2) and (2, 1) are ruled out as inefficient.

As the next step the two players have to decide which of the two efficient strategy pairs they shall use. If they agree to make the decision by tossing a coin, it is easy to see that each player will receive an expected payoff of 7.5. This may appear as a fair solution, acceptable to both parties, but it is not the only possible solution. The players may for instance argue about the random device which shall be used for the final decision. Each player will then argue in favor of the random device which gives the highest probability to the efficient strategy pair most advantageous to him. This clearly leads to a situation similar to those discussed in Chapter VII, i.e. to determine a particular Pareto optimal arrangement as _the_ solution to our problem.

If the two players fail to agree on the random device which shall be used for the final decision, they may consider playing the game in a non-cooperative manner - "each for himself." If they use their first strategy with probabilities x and y respectively, the payoffs will be:

$$M_1(x, y) = 10xy + 5(1 - x)(1 - y) = 5\{1 - x - (1 - 3x)y\}$$

$$M_2(x, y) = 5xy + 10(1 - x)(1 - y) = 5\{2 - 2y - (2 - 3y)x\}$$

From these expressions we see that by choosing $x = 1/3$, Player 1 can secure an expected payoff of $10/3$ for himself, regardless of what Player 2 does. Player 2 can in the same way make his expected payoff equal to $10/3$ by choosing $y = 2/3$.

10.7 In the economic situations which we want to discuss, it is natural to assume that the parties involved can negotiate, and that

contracts are fulfilled. This means that the "Battle of the Sexes" is the more relevant of the two examples which we have discussed. "The Prisoner's Dilemma" is as we have noted trivial when considered as a cooperative game.

Let us now return to the general n-person game, and assume that the n players meet and discuss how they should coordinate their decisions to their mutual benefit. Each player will then argue for an n-tuple of decisions (pure or mixed strategies) which is favorable to him. He may, of course, use any argument which he believes that the other players will swallow. It seems, however, that in an assembly of rational people, he can hope to achieve something only by advancing arguments of the following two types:

(i) He can threaten to refuse to cooperate, i.e. he can threaten to choose his own strategy without any regard to the wishes of the other players, and without informing them about his choice.

(ii) He can appeal to some general principle of fairness or ethics, which he thinks may be acceptable to the other players.

From this it follows that we cannot solve a cooperative game problem without considering the corresponding non-cooperative game. This means that non-cooperative games in some sense are more basic than cooperative games. Any player can refuse to cooperate, and the payoff which he can obtain by this refusal must be an element of his "bargaining power" in the cooperative game.

10.8 The most intriguing problem in bargaining theory is to formulate the "general principle," acceptable to rational persons, as to how such conflicts of interest shall be settled. Since the rationality assumptions

alone do not give a unique solution to the problem, it is obvious that we need some additional assumption about the behavior of the players.

To illustrate the point, we shall return to the example discussed in Chapter VII. In Table 2 of para 7.22 we found the Pareto optimal arrangements which could be reached by an exchange of common stock in the two businesses. Ignoring that these arrangements are not really optimal (because of the unnecessary restriction that only common stock can be exchanged), we can seek an assumption which will single out one particular arrangement which two rational persons can be expected to agree upon.

In para 7.12 we assumed that the two persons behaved as they should according to classical economic theory, and we found that this determined a unique arrangement which gave the two persons the utilities

$$U_1 = 3.66 \quad \text{and} \quad U_2 = 3.58$$

It is, however, possible to introduce a number of alternative assumptions. Let us for instance assume that our two persons for some reason have agreed that the two businesses have the same "intrinsic value" (since they offer the same <u>expected</u> profit), so that in all fairness the common stock must be traded in the ratio 1 : 1.

In para 7.22 we found that the Pareto optimal arrangements were determined by the relations

$$x = \frac{48 - 41k}{7k + 14} \qquad \text{and} \qquad y = \frac{16 - 9k}{7k + 14}$$

Here x and y are the shares which Person 1 owns of Business 1 and 2 respectively. With our new additional assumption, we obviously have $x + y = 1$. This determines a unique arrangement, giving the two persons the utilities

$$U_1 = 3.54 \quad \text{and} \quad U_2 = 3.72$$

The corresponding chares are

$x = 0.598$ and $y = 0.402$

10.9 The additional assumption which we introduced in the preceding paragraph may seem very artificial. The assumption is, however, equivalent to a principle which is widely applied in reinsurance. When two friendly insurance companies of approximately the same size conclude a reciprocal reinsurance treaty, they will usually agree at the outset that the exchange of portfolios shall be on a "net premium basis " This means that they agree to consider only the expected profit on the portfolios which they exchange under the treaty. The implications of an agreement of this kind have been discussed in some detail by Borch [2] and [3].

Our example shows that bargaining over the arrangement which the parties should agree upon can be replaced by bargaining over the general principle which should be applied to settle the conflict situation. It seems that this may often happen in practice. In our example it is quite likely that Person 1 will argue for settling the conflict by the rules of free competition and by applying the ideas of classical economic theory, since these will lead to an arrangement favorable to him. Person 2 may, on the other hand, argue in favor of some "fair price" principle, since this will give him a better deal.

In bargaining between labor and management, principles like "Equal pay for equal work" or "Payment based on productivity" may lead to very different settlements, and either party may argue for the principle which best serves its own interests.

A general principle for resolving conflict situations should be acceptable to all parties. We can formalize this by requiring that the principle should be acceptable as fair to us before we know which part we

are going to play in the game. Ideas of this kind are behind much of our legislation. We consider a law as "good" if we can accept it no matter how we may come in contact with it, as buyer or seller, as tenant or landlord, as creditor or debtor, etc.

10.10 We shall now discuss a general principle proposed by Nash [12], which gives a unique solution to the two-person bargaining problem.

Nash lays down the following four conditions which an arrangement must satisfy to qualify as a "solution" to the problem:

(i) A solution must be invariant under linear transformations of the utility scale.

(ii) A solution must be Pareto optimal.

(iii) Assume that we have a solution, and reduce the problem by removing some possible arrangements which do not represent a solution. The solution to the original problem must also be a solution to the reduced problem.

(iv) If the problem is completely symmetric, the solution is to divide the gain obtained by cooperation equally between the two parties.

Nash then proves that the only arrangement which satisfies these four conditions is the arrangement which maximizes the <u>product</u> of the gains in utility which the two persons make by cooperating.

Applied to the example which we have discussed earlier, the Nash principle gives as solution the exchange of common stock (x, y) which maximizes the product

$$\{U_1(x, y) - U_1(1, 0)\}\{U_2(1 - x, 1 - y) - U_2(0, 1)\}$$

This solution will give the two persons the utilities:

$$U_1 = 3.58 \quad \text{and} \quad U_2 = 3.62$$

This result is different from the two "solutions" which we found in para 10.8.

10.11 The Nash conditions can be spelt out with full mathematical precision, and the proof can be carried through in a rigorous manner. We shall not do this, but we shall give a heuristic argument which will illustrate the essential ideas behind the proof.

Let us consider the bargaining situation illustrated by Figure 1.

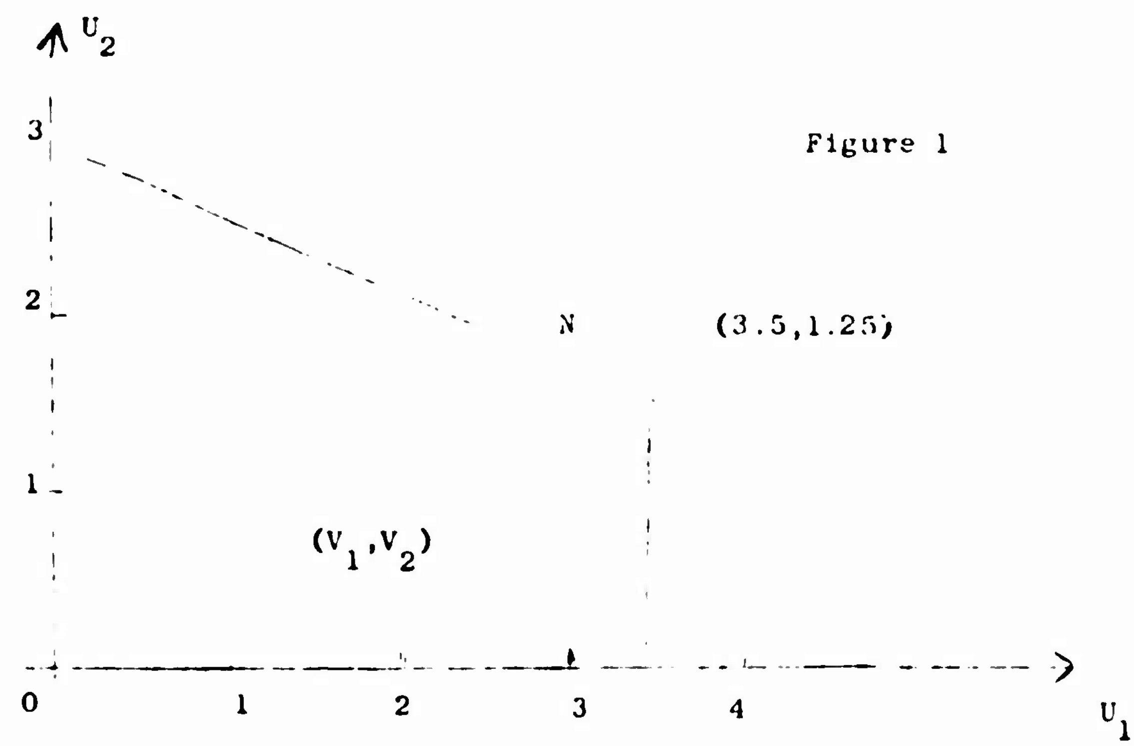

Figure 1

The possible arrangements are represented by the points between the axes and the two lines

$$U_1 + 2U_2 = 6$$

$$U_2 = 3.5$$

The point $(V_1, V_2) = (2, 1)$ represents the utilities which the two persons can secure for themselves by acting in a non-cooperative manner, i.e. by using their mini-max strategies.

The point N, or $(3, 1.5)$ maximizes the product $(U_1 - V_1)(U_2 - V_2)$.

Let us now change the origin on the utility scales so that the point

(V_1, V_2) becomes the origin, i.e. we write $U_1' = U_1 - V_1$ and $U_2' = U_2 - V_2$. This will give us the situation illustrated by Figure 2. The point N has now become $(1, 0.5)$.

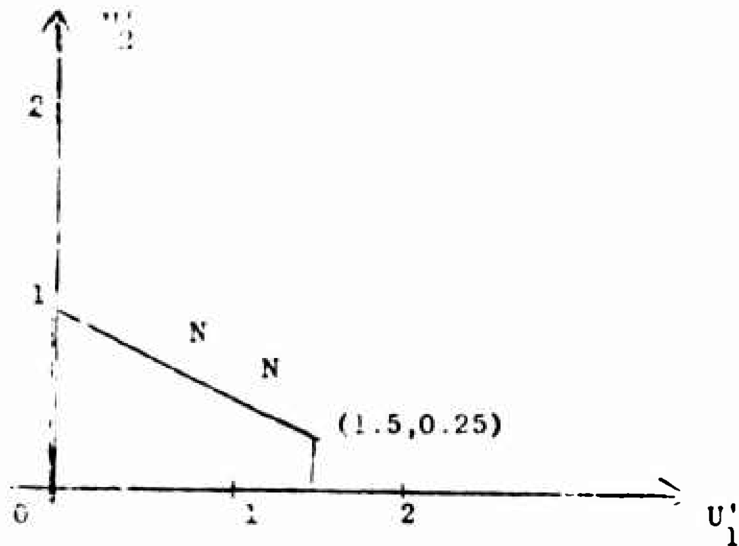

Figure 2

Let us next change the utility scale of Person 2 by the transformation $U_2'' = 2U_2'$. This will give us the situation illustrated by Figure 3. The point N now becomes $(1, 1)$.

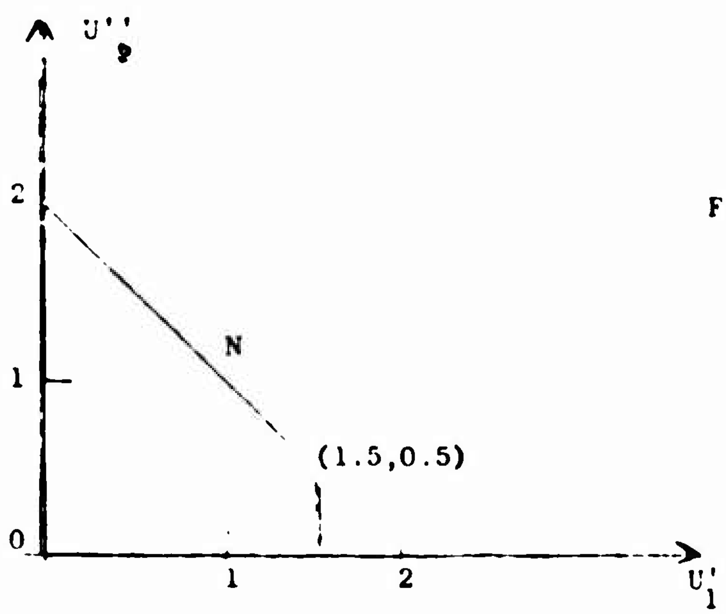

Figure 3

As a last step let us enlarge the game by adding some new possible arrangements at the lower right-hand corner so that we obtain the symmetric situation illustrated by Figure 4.

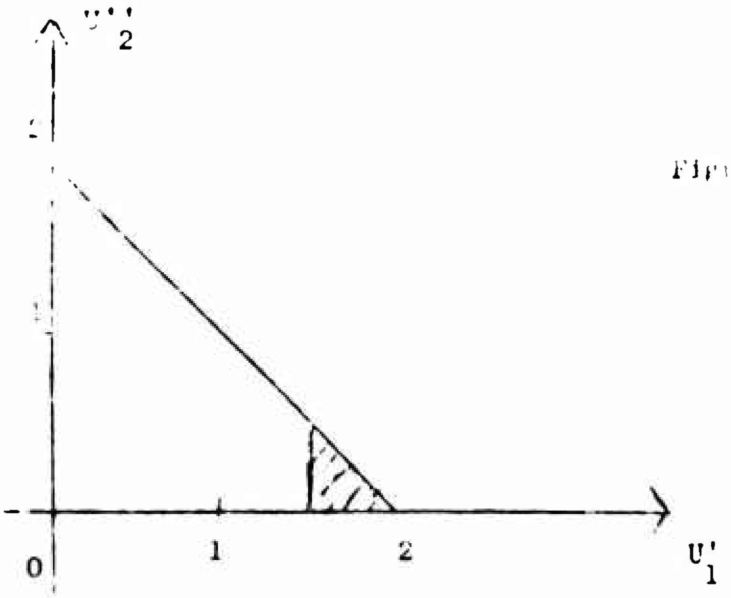

Figure 4

We can now reason backwards. From conditions (ii) and (iv) it follows that the point N represents the unique solution to the game illustrated by Figure 4. From condition (iii) it then follows that N also is the solution to the game illustrated by Figure 3. From condition (i) it further follows that N also represents the solution to the games illustrated by Figure 2 and Figure 1.

10.12 The conditions laid down by Nash are simple, and they look very innocent. At first sight it may appear almost as self-evident that any arrangement which can be seriously considered as a "solution" must satisfy these four conditions, and one may be a little surprised that there exists only one such arrangement. The Nash conditions may, however, in some cases lead to solutions which many people find unreasonable. As an example, let us consider the game illustrated by Figure 5. Here the possible arrangements are represented by the points of a triangle with vertices (0, 0), (1, 1) and (0, 2).

The Nash solution is obviously represented by the point (1, 1). This means, however, that Player 1 gets the highest payoff which he can possibly obtain, whilst Player 2 only gets one half of the maximum gain

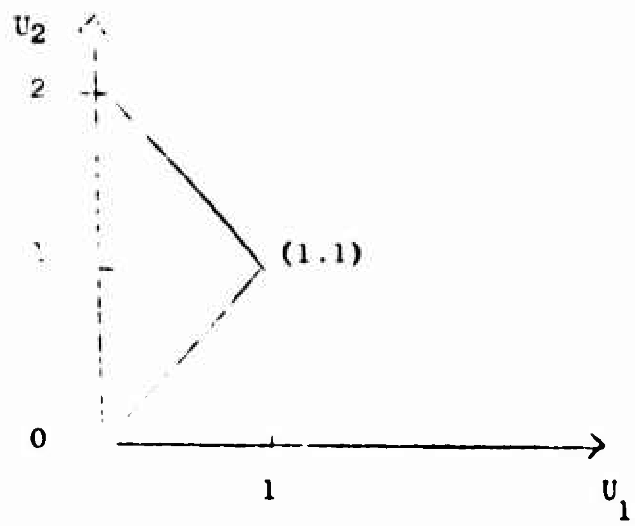

Figure 5

he might have hoped for. If Player 2 argues that this is unfair, he implicitly rejects condition (iii), i.e. that the solution shall be independent of "irrelevant alternatives." If Player 2 presses his argument, he may threaten to refuse to cooperate altogether, and thus reduce the payoff of both players to zero. This means that he is also prepared to reject condition (ii), i.e. that the solution should be a Pareto optimal arrangement.

10.13 The considerations in the preceding paragraphs indicate that we must take some care in specifying what we really mean by a "solution" to a bargaining situation, or more generally to a cooperative game.

It may be tempting to require that a solution should enable us to predict the actual outcome of a bargaining situation. This is, however, clearly asking too much, unless we accept that predictions can only be made in a probabilistic sense. We found already in our discussion of the two-person zero-sum game that the best prediction we could hope to obtain was a probability distribution over the set of all possible outcomes.

As a predictor the Nash solution is not very attractive, since it considers only Pareto optimal arrangements. There is ample evidence that

bargaining situations in real life may lead to non-optimal arrangements. Strikes occur, even if a strike never can be the optimal outcome of a conflict between labor and management.

In the simple example which we discussed in para 10.6 the players could threaten to refuse cooperation, and thus bring about a non-optimal outcome. Such threats cannot be "credible" if we at the outset assume that they will never be carried out in bargaining between rational persons. It seems that if we believe that threats play a real part in bargaining situations, we must accept that the outcome sometimes may be a sub-optimal arrangement.

A solution theory which predicts the outcome of a bargaining situation by specifying a probability distribution over the possible outcomes is a hypothesis which can be tested experimentally. This is a very attractive aspect of any theory, and there exists an extensive literature on so-called "experimental games." We shall not try to summarize this literature. The interested reader may consult a review article by Rapoport and Orwant [15], who discuss 30 different experiments, which constitute a fairly representative sample of such studies made up to the end of 1961. An interesting experiment, not mentioned in this review article, was conducted by **Stone** [17] with students at Stanford University. **Stone** found that the Nash solution did not give very good predictions of the outcome of a series of games, which essentially were of the non-cooperative kind.

10.14 Instead of taking the position of an outsider, trying to predict the outcome, we can look at the bargaining situation from above, as an umpire or arbiter, and draw up the general rules which we will use to resolve the conflict. This leads us to look at the "solution" as an arbitration scheme, i.e. a set of rules which will make it possible for

the arbiter to propose an arrangement which will be accepted as "fair" by all parties. This idea was first formulated by Raiffa [14], but it is inherent in many earlier papers on game theory. An arbitration scheme will in fact give us just the "general principle" or "additional assumption" which we have been chasing earlier in this chapter.

As an arbitration scheme the Nash solution appears very attractive. It seems almost self-evident that an arbitrated arrangement must be Pareto optimal if it shall be accepted by the players (condition (ii)). It is also hard to justify an arbitration scheme which does not lead to a symmetric arrangement when applied to a completely symmetric situation (condition (iv)).

The only difficulty occurs in connection with condition (iii), which we now shall express as follows: "If a possible arrangement, which has been rejected, becomes impossible, this should not change the arbiter's decision." This seems a very reasonable condition, but it led to an arrangement in para 10.12 which some people seem to reject as unfair. The following argument may help to clarify this apparent paradox.

If it is possible for a player to inflict a loss on the opponent, he can threaten to do so. Such threat possibilities will strengthen his bargaining position, and this should be reflected in the arbitrated arrangement, which in a sense must include the compensation the player receives for not carrying out his threats. The Nash solution takes this element into account by referring all gains to the minimum payoff which the player can secure for himself - regardless of what the opponent does.

On the other hand, it is not obvious that the bargaining position of a player should be weakened if it becomes possible for him to do the opponent a "good turn." Condition (iii) states essentially that such

possibilities of exercising charity are irrelevant, and should not be taken into account by an arbiter.

In practice such possibilities may, however, be considered - on vaguely formulated ethical reasons. It is probably more profitable to sue an insurance company with a weak case than to sue one's neighbor with a strong case.

10.15 Let us now return to the general n-person game. From the minimax theorem it follows that Player i can make certain that his expected payoff does not fall under a value $v(\{i\})$, regardless of what the $n-1$ other players do.

Let us assume that Player i and Player j get together and form a coalition which acts as one player in the game against $n-2$ other players. By applying a minimax strategy, this coalition can make sure that its expected payoff does not fall below a certain value $v(\{i\ j\})$.

It is natural to assume that the two players cannot lose by forming a coalition, so that we have

$$v(\{ij\}) \geq v(\{i\}) + v(\{j\})$$

To generalize this idea, let us consider an arbitrary set of players $S \subset N$. Let us assume that these players by forming a coalition can make sure that their joint payoff is at least equal to $v(S)$.

Let us next consider two coalitions, represented by two disjoint subsets R and S of N, and let $v(R \cup S)$ be minimum gain which these two coalitions can obtain by joining forces and acting as one player.

The argument used above leads us to assume

$$v(R \cup S) \geq v(R) + v(S)$$

The function $v(S)$ is called the characteristic function of the game. It is a real-valued super-additive function, defined for all subsets of N. It is clear that the whole strategic structure of the game is contained in the characteristic function.

By introducing this central concept of game theory in such a summary manner, we have swept a number of difficulties under the carpet. It has no meaning to talk about $v(S)$ as the payoff to a coalition unless we make far-reaching assumptions about "inter-personal comparability of utility" and unlimited possibility of "side-payments" within coalitions. We shall however not formulate these assumptions. Most of the results which we shall present can be stated without such assumptions if we are prepared to introduce vector-valued characteristic functions and the more cumbersome notation which this will involve.

10.16 Let us now consider an arbitrary payoff vector $x = (x_1, x_2 \dots x_n)$, i.e. an arrangement which gives Player i the payoff x_i $(i=1,2\dots n)$.

A payoff vector x is called an imputation it satisfies the two conditions:

(i) $x_i \geq v(\{i\})$ for all i

(ii) $\sum_{i=1}^{n} x_i = v(N)$

These two conditions express the individual and collective rationality which we have encountered several times in the preceding chapters. It seems natural to require a payoff vector to be an imputation, if it shall be seriously considered as a potential solution to a bargaining situation. However all imputations do not have equal merits as solutions, and our problem is to eliminate some of the less attractive imputations.

Von Neumann and Morgenstern attack this problem by introducing the concept of dominance

An imputation $y=(y_1 \ldots y_n)$ is said to dominate an imputation $x=(x_1 \ldots x_n)$ if there exists a coalition S (a subset of N) such that

(i) $v(S) \geq \sum_{i \in S} y_i$

(ii) $y_i > x_i$ for all $i \in S$

An imputation which is dominated via some set of players S does not seem very acceptable as a solution, since these players can do better by forming a coalition and "go it alone".

Von Neumann and Morgenstern define the Solution to the game as the set A of imputations which satisfy the following two conditions:

(i) No imputation in A is dominated by another imputation in A.

(ii) Every imputation not in A is dominated by some imputation in A.

To illustrate the meaning of this Solution concept, we shall apply it to a few simple examples.

10.17 As a first illustration let us consider a two person game of the type we discussed in paragraph 10.6. Here the imputations are the payoff vectors (x_1, x_2) which satisfy the conditions

$$x_1 + x_2 = v(\{1,2\}) = 15$$

$$x_1 \geq v(\{1\}) = 10/3$$

$$x_2 \geq v(\{2\}) = 10/3$$

Here there can be no dominance, so the solution consists of all imputations, or in our previous terminology, of all Pareto-optimal arrangements.

As our next example, let us consider a three-person game defined by the characteristic function

$$v(1) = v(2) = v(3) = 0$$

$$v(1,2) = v(1,3) = v(2,3) = 1$$

$$v(1,2,3) = 1$$

We shall first consider the set F consisting of the three imputations

$$(\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

$$(\tfrac{1}{2}, 0, \tfrac{1}{2})$$

$$(0, \tfrac{1}{2}, \tfrac{1}{2})$$

and prove that this set is a solution.

Let us consider an arbitrary imputation (x_1, x_2, x_3).

From the defintion it follows that

$$\sum_{i=1}^{3} x_i = 1 \ , \quad x_i \geq 0$$

It is obvious that either:

(i) 2 elements in this vector is equal to $\frac{1}{2}$, and the third is zero or:

(ii) 2 elements are smaller than $\frac{1}{2}$.

In the former case our imputation belongs to our initial set F.

In the latter case our imputation is dominated by one of the three imputations in the initial set F.

It is easy to see that none of the three imputations dominates the two others. Hence the set F constitutes a solution.

1.18 Let us next consider the set $F_1(c)$ of all imputations of the form

$$(c, x_2, x_3)$$

where c is a non-negative constant.

It is obvious that

$$x_2 + x_3 = 1 - c$$

and that no imputation if $F_1(c)$ dominates any of the other imputations in the set.

Let us then consider an imputation (y_1, y_2, y_3) not in $F_1(c)$.

If $c > y_1$ we have $x_2+x_3 < y_2+y_3$

y_2 and y_3 cannot both be greater than $\frac{1}{2}$.

If $c < \frac{1}{2}$, we can take either x_2, or x_3 greater than $\frac{1}{2}$, and hence find an imputation (c, x_2, x_3) in $F_1(c)$ which dominates (y_1, y_2, y_3)

If $c < y_1$, we have $x_2+x_3 > y_2+y_3$.

In that case we find an imputationin $F_1(c)$ which dominates (y_1, y_2, y_3) by taking $x_2 > y_2$ and $x_3 > y_3$

From these considerations it follows that the Solution consists of the following four sets of imputations.

(i) $F : (\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{1}{2}, 0, \frac{1}{2})$ and $(0, \frac{1}{2}, \frac{1}{2})$

(ii) $F_1(c) : (c, x_2, x_3)$

(iii) $F_2(c) : (x_1, c, x_3)$ $\quad 0 \leq c < \frac{1}{2}$

(iv) $F_3(c) : (x_1, x_2, c)$

10.19 The results of the preceding paragraph mean that any imputation in our simple three-person game belongs to a set, which according to the definition of von Neumann and Morgenstern is a Solution to the game. This is not a very satisfactory conclusion for our purpose. We set out to find one payoff vector which we could accept as the unique solution to a bargaining situation. It is then directly disturbing to find that practically every payoff vector may have a claim to be accepted as a solution.

It would however be premature to launch any general criticism against the solution concept of von Neumann and Morgenstern from this

basis. Most of the games which have been investigated so far, have, like our simple example, an embarrassingly large number of solutions, but it has so far not been possible to prove that <u>every</u> game has a solution, i.e. that there exists in every game a non-empty set A of imputations which satisfies the two conditions in paragraph 10.16.

Von Neumann and Morgenstern argue with strength and at length that only the set A in its entirety can be considered as the solution. We shall not try to summarize these arguments, but we shall endorse the following quotation from another book on game theory:

"The full flavor of their argumement is hard to recapture, and it can only be recommended that the reader turn to the discussions of solutions in their book" ([10] p. 206).

10.20 The essential idea behind the solution concept of von Neumann and Morgenstern is that an imputation must have some stability properties to be acceptable as a solution. To illustrate this, let us again consider the set of imputations F, which we studied in paragraph 10.17.

Let us assume that the players by some process have arrived at the payoff vector:

$$(\frac{1}{2}, \frac{1}{2}, 0)$$

This is not very satisfactory to Player 3, and he may approach Player 2 and propose the imputation

$$(0, \frac{3}{4}, \frac{1}{4})$$

This dominates the original imputation - via coalition (2,3)

If Player 2 accepts this, Player 1 will be the dissatisfied one. He may then approach Player 3, and propose the imputation:

$$(\frac{1}{2}, 0, \frac{1}{2})$$

which dominates $(0, \frac{3}{4}, \frac{1}{4})$ via coalition (1, 3). Player 3 will in all probability accept this.

Now Player 2 may approach any of the other players and propose imputations, such as

$$(\frac{3}{4}, \frac{1}{4}, 0) \quad \text{or} \quad (0, \frac{1}{4}, \frac{3}{4})$$

However these two players should now have learned by observing the fate of Player 2. He was greedy, and tried to obtain more than the solution allocated to him, and as a result he lost everything. Hence the imputation $(\frac{1}{2}, 0, \frac{1}{2})$ may remain stable.

Von Neumann and Morgenstern suggest that the constant c which occurred in the other solutions $F_1(c)$, $F_2(c)$ and $F_3(c)$ may be interpreted as a measure of the amount of discrimination which society will permit. If unlimited discrimination is allowed, we have c=0, and the solution consists of the three imputations in F.

10.21 In paragraph 10.16 we defined an imputation as a payoff vector $x = (x_1 \ldots x_n)$ which satisfied the two conditions:

(i) $\quad x_i \geq v(i) \quad$ for all i

(ii) $\quad \sum_{i=1}^{n} x_i = v(N)$

The first of these conditions state that no player will accept less than he can obtain by acting along - against the n-1 other players.

It is tempting to assume that each coalition will exercise the same degree of rationality as an individual player.

This leads us to lay down a third condition

(iii) $\quad \sum_{i \in S} x_i \geq v(S) \quad$ for all S in N

The set of all imputations which satisfy these three conditions is called the Core of the game, a concept first introduced by Gillies [4].

At first sight the core appears to be a very attractive concept - just the answer to our search for a device which will cut down the number of imputations which may be considered as potential solutions to our problem. The trouble is however that the core often does this job too drastically. In a number of games the core is empty, i.e. there exists no imputation which satisfies the three conditions.

In the three-person game which we discussed in paragraph 10.17 the core is the set of vectors (x_1, x_2, x_3) with non-negative elements satisfying the four conditions:

$$x_1 + x_2 \geq v(1,2) = 1$$

$$x_1 + x_3 \geq v(1,3) = 1$$

$$x_2 + x_3 \geq v(2,3) = 1$$

$$x_1 + x_2 + x_3 = v(1,2,3) = 1$$

Adding the first three conditions we obtain

$$x_1 + x_2 + x_3 \geq 3/2$$

which obviously contradicts the fourth condition.

If the core is non-empty, i.e. if we have

$$v(1,2) + v(1,3) + v(2,3) \leq 2v(1,2,3)$$

we can obtain the following interval for the payoff to Player 1

$$v(1) \leq x_1 \leq v(1,2,3) - v(2,3)$$

The right-hand inequality states that Player 1 cannot obtain more than what he contributes by joining a coalition of the two other players.

10.22 The contribution which a player can make by joining a coalition appears on intuitive reasons to be an essential element of his bargaining power.

To develop this idea let us consider an arbitrary coalition S of s players, and assume Player i, who was not in S joins the coalition. The contribution which Player i makes to the total payoff is

$$v(S \cup \{i\}) - v(S)$$

Let us now assume that the characteristic function is such that an all-player coalition must be formed in order to obtain the maximum total payoff. Let us further assume that this n-player coalition is formed by the players joining the coalition one at the time, i.e. by a successive buildup of a 2, 3, 4 ... player coalition.

The n players can be ordered in n! different sequences. Hence the n-person coalition can be formed in n! different ways.

The s players which are in the coalition S before Player i joins can be arranged in s! different ways. The n-s-1 players which join the coalition after Player i can be arranged in (n-s-1)! different ways.

If all the n! ways in which the n-player coalition can be formed are equally probable, there will be a probability

$$\frac{s!(n-s-1)!}{n!}$$

that Player i shall join the coalition S.

Let us now assume that if this should happen, the payoff to Player i will be exactly his contribution $v(S \cup \{i\}) - v(s)$. This means that the expected payoff to Player i is

$$\varphi_i = \sum_{S \text{ in } N} \frac{s!\,(n-s-1)!}{n!} \{v(S \cup \{i\}) - v(s)\}$$

where the sum is taken over all subsets on N .

φ_1 is called the Shapley-value of the game to Player 1.

The expression above was derived by Shapley [13] via an approach quite different from the one we have taken. His starting point was to determine the value a player should assign to his right to participate in a game.

10.23 We shall now compute the Shapley value for a few simple games.

For n=2 there are only four subsets: $\emptyset$, {1}, {2} and {1,2}, so that the Shapley-value to Player 1 is:

$$\varphi_1 = \frac{1}{2}\{v(1.2) - v(2)\} + \frac{1}{2}\{v(1) - v(\emptyset)\}$$

$$= \frac{1}{2}\{v(1.2) - v(2) + v(1)\}$$

For Player 2 we find

$$\varphi_2 = \frac{1}{2}\{v(1,2) - v(1) + v(2)\}$$

If the game is zero-sum, we have $v(1,2) = 0$ and $v(1) = -v(2)$, so that the expressions reduce to

$$\varphi_1 = v(1) \quad \text{and} \quad \varphi_2 = -v(1)$$

This is the mini-max solution to the game.

The Nash-solution to the game is the payoff vector (x_1, x_2) which maximizes the product

$$\{(x_1 - v(1)\}\{x_2 - v(2)\}$$

subject to the condition

$$x_1 + x_2 = v(1.2)$$

If we solve this maximizing problem, we find

$$x_1 = \varphi_1 \quad \text{and} \quad x_2 = \varphi_2$$

For the three-person game which we have discussed earlier, we find:

$$\varphi_1 = \frac{2}{6}\{v(1,2,3) - v(2,3)\} + \frac{1}{6}\{v(1,2) - v(2)\} + \frac{1}{6}\{v(1,3) - v(3)\} = \frac{1}{3}$$

Since the game is symmetric, the Shapley value of the two other players is also equal to $\frac{1}{3}$.

These examples show that the Shapley value contains the mini-max solution and the Nash solution as special cases. It does however not agree with the von Neumann-Morgenstern solution to the three-person game, although the Shapley value is equal to the average payoff of the three imputations in the solution set F.

10.24 The results in the preceding paragraphs seem to indicate that the Shapley value may be taken as the solution to our problem.

As an <u>arbitration scheme</u> the Shapley value appears very reasonable. It seems natural that an arbiter should consider the contributions which a player can make to all possible coalitions, and then assign him a payoff which is a weighted average of these potential contributions. The Shapley value obviously leads to a Pareto optimal arrangement, and if the game has a non-empty core, the Shapley value lies in the core.

As a <u>predictor</u> the Shapley value is less attractive, mainly because it is crude. The Shapley value can be interpreted as the expected payoff to a player, and an ideal predictor should give us the probability distribution over all possible payoffs - not just the first moment of this distribution.

In our three-person game the imputation consisting of expected payoffs was $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, and this did not have the stability properties which we would like to require of an acceptable solution. Any two players could form a coalition and bring about an imputation of the type ([illegible], [illegible], 0), which has the required stability. However the imputation $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ appears to be the only reasonable arbitrated outcome of a symmetric three-person game.

10.25 Our findings so far indicate that we somehow must make a choice. It seems impossible to find a solution concept which at the same time has all the desirable stability properties, and also appears as a reasonable arbitrated outcome of the game.

An attempt to capture the best of both worlds has been made by Aumann and Maschler [1] who introduced the ingenious, and mathematically very elegant concept of the Bargaining Set.

The Bargaining Set includes:

(i) The core, if the game has a non-empty core

(ii) The equilibrium points corresponding to the set F in the three-person game which we have used as an illustration.

Some experiments by Maschler [11] with high school students in Jerusalem, indicate that the Bargaining Set gives a fairly good prediction of the outcome of bargaining situations. Aumann and Maschler have not so far been able to offer any suggestions as to the relative importance of the imputations in the core and the imputations which satisfy the equilibrium conditions. This is however not surprising, since the relative importance of these two considerations apparently must depend on the particular environment in which the bargaining takes place.

10.26 We cannot conclude this chapter without giving at least some references to the work of Harsanyi, which is much too rich and varied to be discussed in detail.

Harsanyi's first contribution [5] was to show that the solutions to the bargaining problem suggested by Hicks and Zeuthen in terms of classical economic theory, were equivalent to the Nash solution.

Later he has developed a number of general bargaining models [6] and [7], which i.a. contain the Shapley value as a special case, when inter-person comparisons of utility are possible.

REFERENCES

[1] AUMANN, R. and M. MASCHLER: "The Bargaining Set for Cooperative Games", Annals of Mathematics Studies, No. 52, pp. 443-476, Princeton, 1964.

[2] BORCH, K.: "Reciprocal Reinsurance Treaties", The ASTIN Bulletin, 1960 pp. 170-191.

[3] BORCH, K.: "Equilibrium in a Reinsurance Market", Econometrica, 1962 pp 424-444.

[4] GILLIES, D.: "Solutions to General Non-zero-sum Games", Annals of Mathematics Studies No. 40, pp 47-85, Princeton, 1959.

[5] HARSANYI, J.: "Approaches to the Bargaining Problem before and after the Theory of Games", Econometrica, 1956, pp. 144-157.

[6] HARSANYI, J.: "A Bargaining Model for the Cooperative n-person Game", Annals of Mathematics Studies, No. 40, pp.325-355, Princeton, 1959.

[7] HARSANYI, J.: "A Simplified Bargaining Model for the n-person Cooperative Game", International Economic Review, 1963, pp. 194-220.

[8] LAVE, L.: "An Empirical Approach to the Prisoner's Dilemma", Quarterly Journal of Economics, 1962, pp. 424-436.

[9] LAVE, L.: "Factors Affecting Co-operation in the Prisoner's Dilemma", Behavioral Science, 1965, pp. 26-38.

[10] LUCE, R. D. and H. RAIFFA: Games and Decisions, John Wiley & Sons, 1957.

[11] MASCHLER, M.: "Playing an n-person Game - an Experiment", Research Memorandum No. 73, Econometric Research Program, Princeton University, 1965.

[12] NASH, J.: "The Bargaining Problem", Econometrica, 1950, pp. 155-162.

[13] NEUMANN, J. von and O. MORGENSTERN, Theory of Games and Economic Behavior, Princeton University Press, 1944.

[14] RAIFFA, H.: "Arbitration Schemes for Generalized Two-person Games", *Annals of Mathematics Studies*, No. 28, pp. 361-387, Princeton, 1953.

[15] RAPOPORT, A. and C. ORWANT: "Experimental Games: A Review", *Behavioral Science*, 1962, pp. 1-37.

[16] SHAPLEY, L. S.: "A Value for n-person Games", *Annals of Mathematics Studies*, No. 28, pp. 307-317, Princeton, 1953.

[17] STONE, J.: "An Experiment in Bargaining Games", *Econometrica*, 1958, pp. 286-296.